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LETTER TO THE EDITOR

Arithmetical polynomials derived from the spectrum of cyclic permutation operators on spin spaces

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Abstract. Novel arithmetical polynomials in x of degree n congruent to zero to modulus n satisfy several interesting identities and offer a natural description for spectra of cyclic permutation operators on n x-component Potts spins spaces.

Solutions of spin models on a lattice, which involve a cyclic permutation operator in spin spaces, could well be strongly dependent on arithmetical properties such as primality factorisation of the number of spins present in the system. First hints of this surprising fact showed up in our recent work (Audit and Truong 1989) on a Hubbard chain with a hole; but, in our opinion, it could be widespread, though hitherto unnoticed, because of the rarity of exact analytic solutions for finite systems and of difficulties encountered in extending to numerical calculations of large-sized systems. The origin of the phenomenon rests on peculiarities of cyclic permutation operator spectra, that are interpreted in this letter in terms of some novel arithmetical polynomials involving in their structure both integers n and x, i.e. the essential characteristic features of a system comprising n x-component Potts spins. So matters pertaining as well to spin and number theories are entangled in this problem, whose solution yields interesting results relevant to both domains.

Consider first the vector space spanned by n-fold tensor products of x-component Potts spin states; there are x^n values of all the spins denoted $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_n)$ and each σ_i takes the values $1, 2, \ldots x$. A representation of a cyclic permutation operator in this space is the $x^n \times x^n$ matrix

$$(\boldsymbol{G}_n)_{\boldsymbol{\sigma}\boldsymbol{\sigma}'} = \delta(\boldsymbol{\sigma} - \boldsymbol{m}_n\boldsymbol{\sigma}') \tag{1}$$

where $\delta(\boldsymbol{\sigma})=1$ (respectively 0) if $\boldsymbol{\sigma}=0$ (respectively $\neq 0$) and $\boldsymbol{m}_n$ is the $n\times n$ cyclic matrix (Audit 1985)

$$(\boldsymbol{m}_n)_{ii'} = \delta_{i+1,i'} \qquad (\mathrm{mod}\ n).$$

The matrix $\boldsymbol{G}_n$ is semisimple and also n-potent $\boldsymbol{G}_n^{p+n}=\boldsymbol{G}_n^p$; it has n eigenvalues ω_n^α, $\alpha=0,1,\ldots,n-1$, equal to the nth roots of unity. In a calculation, to be detailed in a forthcoming publication, we have found the multiplicities $m(\omega_n^\alpha)$ of the eigenvalues to be expressible in terms of the novel arithmetical polynomials

$$A_\alpha(n,x) = \sum_{d|n} c(\alpha,d)x^{n/d} \tag{2}$$

whose coefficients are the Ramanujan sums

$$c(\alpha,d) = \sum_{\substack{1\leqslant l\leqslant d\\(l,d)=1}} \omega_d^{\alpha l}$$

where, as usual in number theory, $d|n$ denotes that d is a divisor of n and (l, d) is the greatest common divisor of l and d. The Ramanujan sums have integer values (Hardy and Wright 1938), so that the polynomials $A_\alpha(n, x)$ are arithmetical functions in n.

Two special cases of those polynomials, for $\alpha = 0$ and 1, are connected with two of the most important arithmetical functions as follows: since $c(0, d) = \phi(d)$ is the Euler function, hence

$$A_0(n, x) = \sum_{d|n} \phi(d) x^{n/d} \tag{3}$$

and since $c(1, d) = \mu(d)$ is the Möbius function, hence

$$A_1(n, x) = \sum_{d|n} \mu(d) x^{n/d} \tag{4}$$

and by applying the Möbius inversion formula we have the following identity:

$$x^n = \sum_{d|n} A_1(d, x). \tag{5}$$

Another very interesting identity is obtained from the orthogonal properties of Ramanujan sums proved by Carmichael (1932); we obtain

$$x^{n/d} = \frac{1}{n\phi(d)} \sum_{\alpha=0}^{n-1} c(\alpha, d) A_\alpha(n, x). \tag{6}$$

Turning now to the spectrum of G_n, the multiplicities of the eigenvalues are merely given by

$$m(\omega_n^\alpha) = n^{-1} A_\alpha(n, x) \tag{7}$$

that must be, by definition, an integer; therefore we have the polynomial congruences of identical degree and modulus

$$A_\alpha(n, x) \equiv 0 \qquad (\mathrm{mod}\ m) \tag{8}$$

and since $A_\alpha(n, x) = A_{n-\alpha}(n, x)$, there are $[n/2]+1$ distinct relations obtained for $\alpha = 0, 1, \ldots, [n/2]$. Those polynomial congruences are reduced to the binomial congruence of Fermat when n is prime; thus here we have a generalisation of Fermat's little theorem to an arbitrary integer.

Moreover, it can be shown that the pth power of G_n has a trace in the simple form

$$\mathrm{Tr}\, G_n^p = x^{(p,n)} \qquad p \leqslant n \tag{9}$$

that can be identified with an expression in terms of the multiplicities (7) yielding the identity

$$x^{(p,n)} = n^{-1} \sum_{\alpha=0}^{n-1} A_\alpha(n, x) \omega_n^{\alpha p}. \tag{10}$$

Finally, let us consider an example and calulate a few of the polynomials explicitly; for $n = 6$ we have

$$A_0(6, x) = x^6 + x^3 + 2x^2 + 2x$$

$$A_1(6, x) = A_5(6, x) = x^6 - x^3 - x^2 + x$$

$$A_2(6, x) = A_4(6, x) = x^6 + x^3 - x^2 - x$$

$$A_3(6, x) = x^6 - x^3 + 2x^2 - 2x$$

and one can easily check that for any positive integer x, according to (8), those four polynomials are divisible by 6. We can verify the identity (10); for $p=n=6$ we have

$$x^6=\frac{1}{6}\sum_{\alpha=0}^{5} A_\alpha(6,x).$$

The identity (6) is also satisfied, as we have for $d=3$

$$x^2=\frac{1}{12}\sum_{\alpha=0}^{5} 2\cos\frac{2\pi\alpha}{3} A_\alpha(6,x)$$

and, according to (5),

$$x^6=A_1(6,x)+A_1(3,x)+A_1(2,x)+A_1(1,x)$$

with

$$A_1(3,x)=x^3-x \qquad A_1(2,x)=x^2-x \qquad A_1(1,x)=x.$$

In conclusion, we now have good knowledge of the arithmetical polynomials of degree n congruent to zero to modulus n, which control through (7), (9) and (10) the spectra of cyclic permutation operators on spin spaces and thereby the main properties of finite-sized models involving such operators, as do the Ising model (Audit 1987), the Hubbard model with a hole (Audit and Truong 1989) and some others currently being investigated.

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